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Transformation Properties of the Lagrangian and Eulerian Strain Tensors

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ARL-TR-908 April 2002

Approved for public release; distribution unlimited.

Army Research Laboratory

Adelphi, MD 20783-1197

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Abstract

A coordinate independent derivation of the Eulerian and Lagrangian strain tensors of finite deformation theory is given based on the parallel propagator, the world function, and the displacement vector field as a three-point tensor. The derivation explicitly shows that the Eulerian and Lagrangian strain tensors are two-point tensors, each a function of both the spatial and material coordinates. The Eulerian strain is a two-point tensor that transforms as a second rank tensor under transformation of spatial coordinates and transforms as a scalar under transformation of the material coordinates. The Lagrangian strain is a two-point tensor that transforms as a scalar under transformation of spatial coordinates and transforms as a second rank tensor under transformation of the material coordinates. These transformation properties are needed when transforming the strain tensors from one frame of reference to another moving frame.

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1. Background

The U.S. Army is developing an electromagnetic gun (EMG) for battlefield applications. During the past few years, on a recurring basis, Dr. John Lyons (ARL Director) and Dr. W. C. McCorkle (Director of U. S. Army Aviation and Missile Command) have requested that I look at some of the physics of the EMG. In the most recent request, I was asked to look at stresses in a rotating cylinder. For an elastic cylinder, this is a classic problem that is solved in many texts on linear elasticity [1–6]. However, when these derivations are examined closely, one finds certain shortcomings [7]. Therefore, I spent some time looking at the problem of stresses in elastic rotating cylinders, which resulted in a manuscript [7]. In the course of this work [7], I had to clearly understand the transformation properties of the Largangian and Eulerian strain tensors of finite deformation theory. I was quite dissatisfied with the standard derivations of the Largangian and Eulerian strain tensors because these derivations take either of two (both unpalatable) approaches. The first approach uses shifter tensors, which are often defined as inner products between two basis vectors at *two different spatial locations* [8,9]. In this approach, basis vectors are not parallel transported to the same spatial location before the inner product is carried out. This is unpalatable, even in Euclidean space, unless one is using Cartesian coordinates. The second approach uses convected (moving) coordinates, and vectors and tensors are associated with a given *coordinate* in the convected (moving) coordinate system, rather than being associated with a point in the underlying space.

In the derivation that I present below, I avoid both of the unpalatable features mentioned above. I provide a coordinate independent derivation of the Lagrangian and Eulerian strain tensors based on standard concepts in differential geometry: the parallel propagator, the world function, and the displacement vector field as a three-point tensor.

The derivation that I present below is also useful for gaining a basic understanding of the role of the unstrained state, or reference configuration, in the definition of the strain tensors. Having a firm conceptual grasp of the role of the unstrained state in the definition of the strain tensors is imperative for understanding the behavior of pre-stressed materials under finite deformations in high-stress applications, such as, for example, in the electromagnetic rail gun [10].

2. Introduction

The theory of stresses in rotating cylinders and disks is of great importance in practical applications such as rotating machinery, turbines and generators, and wherever large rotational speeds are used. In a previous work [7], I gave a detailed treatment of stresses in a rotating elastic cylinder. This is a classic problem that is treated in many texts on linear Elasticity theory [2–6]. These treatments linearize the strain tensor in the gradient of the displacement field, assuming that these (dimensionless) gradients are small. For large angles of rotation, the quadratic terms (in displacement gradient in the definition of strain) are as large as the linear terms, and consequently, these quadratic terms cannot be dropped [7]. In Ref [7], I provide an alternative derivation of stresses in an elastic cylinder that relies on transforming the problem from an inertial frame (where Newton's second law is valid) to the co-rotating frame of the cylinder, where the displacement gradients are small. During the course of that solution, I had to transform the Lagrangian and Eulerian strain tensors of finite elasticity to the (non-inertial) co-rotating frame of reference of the cylinder, which is a moving, accelerated frame. This work required the detailed understanding of the transformation properties of the Lagrangian and Eulerian strain tensors.

The standard derivation of these strain tensors is done with the help of shifter tensors [8,9]. Shifter tensors are often defined in terms of inner products of basis vectors that are located at two different spatial points [8,9]. For me, inner products between vectors at two different points is an unpalatable operation, even in Euclidean space. In order to compute the inner product between two vectors, the vectors must first be parallel transported to the same spatial point (unless we are using Cartesian coordinates, in which case the derivation becomes coordinate specific).

In other treatments, where shifter tensors are not employed in the derivation of strain tensors, convected (moving) coordinates are used, see for example [11–14]. When using convected (moving) coordinates, the coordinates of the initial undeformed point and the deformed point are the same, but the basis vectors change during deformation. In derivations of strain tensors using convected coordinates, vectors and tensors are associated with a given point in the convected (moving) coordinate system, rather than being associated with a point in the underlying (inertial) space. Tensors are absolute geometric objects, and they should properly be associated with a point in the underlying space, and not a given coordinate, e.g., in moving coordinates.

In this work, I avoid the unpalatable features of the strain tensor derivation mentioned in the above two paragraphs. I derive the strain tensors using the concept of absolute tensors, where a tensor is associated with a point

in the space, rather than the coordinates in a given (moving) coordinate system. I provide a coordinate independent derivation of the Lagrangian and Eulerian strain tensors, where I keep track of the positions of the basis vectors. The derivation necessarily uses two-point (and three-point) tensors [8,9,15,16]. The derivation is based on standard concepts in differential geometry: the parallel propagator (a two-point tensor), the world function (a two-point scalar), and the displacement vector field (a three-point tensor). This derivation makes clear the transformation properties of the strain tensors under coordinate transformations from one frame of reference to a second frame that is moving and accelerated (with respect to the first frame).

The derivation below of the Eulerian and Lagrangian strain tensors makes the transformation properties (e.g., to a moving frame) clear. Furthermore, this derivation makes the role of the reference (unstrained) configuration more clear in the definition of the strain tensors. Clarifying this role is important for applying finite deformation theory to pre-stressed materials, which are capable of withstanding higher-stress applications, such as in rotating machinery [7,10]. Finally, the derivation presented here allows the generalization of the definition of strain tensors to the realm where general relativity applies [17,18].

3. Geometric Background

In Euclidean space, a vector can be trivially parallel propagated in the sense that after a round trip the vector still points in the same direction. In Riemannian space, the parallel displaced vector is not equal to itself after the round trip parallel displacement. In this sense, in Euclidean space we need not distinguish the position of a vector because "it always points in the same direction under parallel displacement," even though its components may be different from point to point because the basis vectors, onto which we project the vector, point in a different direction from point to point. So, in Euclidean space the parallel displaced (physical) vector (a geometric object) is thought to point in the same physical direction. In Riemannian space, however, the situation is quite different. In Riemannian space, a vector that is parallel displaced will in general point in a different direction. The physical test is to parallel displace the vector along a curve that returns to the starting point. If there is non-zero curvature, as measured by the Riemann curvature tensor, then upon returning to its starting point the vector components will be different than the initial vector components at the starting point. So, in Reimannian space, it is imperative to specify the position of a vector. In Euclidean space, appropriate to material deformations, I also keep track of the position of a vector. This additional care in Euclidean space, together with the transformation properties of the world function, leads to a clearer understanding of the transformation properties of the Lagrangean and Eulerian strain tensors, under transformations from one system of coordinates to another that is in relative motion (a moving frame).

This section briefly reviews the fundamental geometric quantities that naturally arise in discussion of deformation, but which are not usually discussed in this context. These quantities are the world function (or fundamental two-point scalar of the the space), the parallel propagator, and the position vector. This section will also serve to define my notation. Each of the quantities mentioned are examples of a class of geometric object known as two-point tensors, which occur naturally in the discussion of deformations. I have found useful discussions of general tensor calculus in Synge and Schild [19] and Synge [16], and discussions oriented toward deformation theory in the appendix by Ericksen in Treusdell and Toupin [15], and in Narasimhan [9], Eringen [8], and Eringen [20]. In particular, discussion of two-point tensors can be found in Synge [16], Ericksen [15], Narasimhan [9] and Eringen [8].

3.1 The World Function

The world function was initially introduced into tensor calculus by Ruse [21,22], Synge [23], Yano and Muto [24], and Schouten [25]. It was

further developed and extensively used by Synge in applications to problems dealing with measurement theory in general relativity [16]. An application of the world function to problems of navigation and time transfer can be found in Ref. [26]. Compared to the enormous attention given to tensors, the world function has been used very little by physicists. Yet, when geometry plays a central role, such as in deformation theory, the world function is helpful to clarify and unify the underlying geometric concepts. The world function is simply one-half the square of the distance between two points in the space. In applications to relativity and four-dimensional space-time, the space-time is often taken as a general (curved) pseudo-Riemannian space [16]. In applications to deformation of materials, we are concerned with a Euclidean three-dimensional space. However, for understanding the transformation properties of displacement vectors and strain tensors, it is helpful to use the concept of world function in a Euclidean three-dimensional space described by curvilinear coordinates x^i, $i = 1, 2, 3$, with a metric g_{ij}, which in general is a function of position.

Consider two points in a general Riemannian space, P_1 and P_2, connected by a unique geodesic path (a straight line in Euclidean space) Γ, given by $x^i(u)$, $i = 1, 2, 3$, where $u_1 \leq u \leq u_2$, and $x^i(u)$ are curvilinear coordinates of the path. The coordinates of point $P_1 = \{x_1^i\}$ and point $P_2 = \{x_2^i\}$. In general, a geodesic is defined by a class of special parameters u', $u \cdots$, that are related to one another by linear transformations $u' = au + b$, where a and b are constants. Here, u is a particular parameter from the class of special parameters that define the geodesic Γ, and $x^i(u)$ satisfy the geodesic equations

$$\frac{d^2x^i}{du^2} + \Gamma^i_{jk}\frac{dx^j}{du}\frac{dx^k}{du} = 0 \tag{1}$$

Using Cartesian coordinates z^k (rather than general curvilinear coordinates x^k) in Euclidean space, the Christoffel symbol $\Gamma^i_{jk} = 0$, and the solution of equation (1) is simply

$$z^\alpha(u) = z_1^\alpha + \frac{u - u_1}{u_2 - u_1}\left(z_2^\alpha - z_1^\alpha\right) \tag{2}$$

where $u_1 \leq u \leq u_2$, $i = 1, 2, 3$ and the Cartesian coordinates of points P_1 and P_2 are z_1^α and z_2^α, respectively. In a general Riemannin space, the world function between point P_1 and P_2 is defined as the integral along Γ in arbitrary curvilinear coordinates x^i by

$$\Omega(P_1, P_2) = \frac{1}{2}(u_2 - u_1)\int_{u_1}^{u_2} g_{ij}\frac{dx^i}{du}\frac{dx^j}{du}\,du \tag{3}$$

The value of the world function has a simple geometric meaning: It is one-half the distance between points P_1 and P_2. Its value depends only on the eight coordinates of the points P_1 and P_2. The value of the world function in equation (3) is independent of the particular special parameter u in the sense that under a transformation from one special parameter u to another, u', given by $u = au' + b$, with $x^i(u) = x^i(u(u'))$ and with a and b constant,

the world function definition in equation (3) has the same form (with u replaced by u').

The world function is *the* fundamental two-point invariant that characterizes the space. It is invariant under independent transformation of coordinates at P_1 and at P_2. For a given space, the world function between points P_1 and P_2 has the same value independent of the coordinates used, which makes it a useful coordinate independent quantity. In Euclidean space, using Cartesian coordinates, the world function has the simple form

$$\Omega(z_1^i, z_2^j) = \frac{1}{2}\,\delta_{ij}\,\Delta z^i\,\Delta z^j \tag{4}$$

where δ_{ij} is the Euclidean metric with only non-zero diagonal components $(+1,+1,+1)$, and $\Delta z^i = (z_2^i - z_1^i)$, $i = 1,2,3$, where z_1^i and z_2^i are the Cartesian coordinates of points P_1 and P_2, respectively. (I use the convention that all repeated indices are summed, unless otherwise stated.)

The world function has a number of interesting properties, see Synge [16]. Calculations of the world function for spaces other than Euclidean spaces, namely four-dimensional space-time, can be found in Refs. [16,26–29]. In what follows, I restrict myself to a three-dimensional space. By transforming to a new system of coordinates, say spherical coordinates,

$$x = r\,\cos\theta\,\cos\phi \tag{5}$$
$$y = r\,\cos\theta\,\sin\phi \tag{6}$$
$$z = r\,\cos\theta \tag{7}$$

the world function in equation (4) can be expressed as a function of spherical coordinates of point $P_1 = (r_1, \theta_1, \phi_1)$, and $P_2 = (r_2, \theta_2, \phi_2)$.

Consider a geodesic given by equation (1) in a general three-dimensional Riemannian space. The covariant derivatives of the world function have two important properties:

$$\frac{\partial\,\Omega(P_1,P_2)}{\partial\,x_2^i} = (u_2 - u_1)\left(g_{ij}\,\frac{dx^j}{du}\right)_{P_2} = L\,\lambda_{i_2} \tag{8}$$
$$\frac{\partial\,\Omega(P_1,P_2)}{\partial\,x_1^i} = -(u_2 - u_1)\left(g_{ij}\,\frac{dx^j}{du}\right)_{P_1} = -L\,\lambda_{i_1} \tag{9}$$

where

$$L = [2\,\Omega(P_1,P_2)]^{1/2} = \int_{P_1}^{P_2} ds = \int_{u_1}^{u_2}\left[g_{ij}\,\frac{dx^i}{du}\,\frac{dx^j}{du}\right]^{1/2} du \tag{10}$$

is the length of the geodesic between P_1 and P_2, g_{ij} is the metric in coordinates x^i, and λ_{i_1} and λ_{i_2} are components of the unit tangent vectors at end points P_1 and P_2 (assuming non-null geodesics [16]):

$$\lambda_{i_1} = \left(g_{ij}\,\frac{dx^j}{ds}\right)_{P_1} \tag{11}$$
$$\lambda_{i_2} = \left(g_{ij}\,\frac{dx^j}{ds}\right)_{P_2} \tag{12}$$

where the relation between parameter u and arc length s is given by equation (10). In equations (8) and (9), the covariant partial derivatives with respect to x^i_1 and x^i_2 are done with respect to the coordinates of points P_1 and P_2. See figure 1.

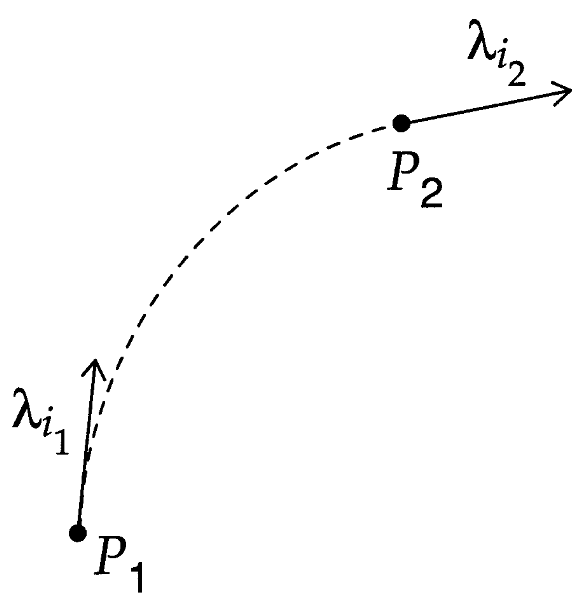
For the special case of interest in deformation of materials, the space is three-dimensional Euclidean, the geodesic is a straight line, and the vectors λ_{i_1} and λ_{i_2} are colinear, although I still consider them as existing at distinct points. Using Cartesian coordinates and the explicit form of the world function in equation (4) and equations (8) and (9) take the form

$$\frac{\partial\,\Omega(P_1,P_2)}{\partial\, z^i_2} = \equiv \Omega_{i_2} = \delta_{ij}\left(z^j_2 - z^j_1\right) = L\,\lambda_{i_2} \tag{13}$$

$$\frac{\partial\,\Omega(P_1,P_2)}{\partial\, z^i_1} = \equiv \Omega_{i_1} = -\delta_{ij}\left(z^j_2 - z^j_1\right) = -L\,\lambda_{i_1} \tag{14}$$

where I used Synge's short-hand notation for the components of the covariant partial derivatives by putting subscripts on the indices to indicate which coordinates were differentiated. This short-hand notation is particularly convenient to show the transformation properties of the world function and to indicate the spatial location of vectors and tensors. For example, the quantity Ω_{i_2} transforms as a vector under coordinate transformations at P_2 and as a scalar under coordinate transformations at point P_1. The quantity λ_{i_2} is a vector located at point P_2. Note that by virtue of their definitions in the left side of equations (13) and (14), the right sides are two-point tensors, whose components are functions of coordinates at point P_1 and P_2. For example, the right side of equation (14) is a product of a two-point scalar L, and a one point vector, λ_{i_1} at P_1.

Figure 1. A geodesic path is shown in three dimensions with tangent unit vector at the ends.



3.2 Parallel Propagator

Given a vector with components v^{i_1} at point P_1, the vector is said to be parallel propagated from P_1 to P_2 along a geodesic curve C specified by $x^i(u)$, $u_1 \leq u \leq u_2$, where $P_1 = \{x^i(u_1)\}$ and $P_2 = \{x^i(u_2)\}$, when its covariant derivative is zero along this curve:

$$\frac{dv^i}{du} + \Gamma^i_{jk} v^j \frac{dx^k}{du} = 0 \tag{15}$$

Equation (15) is a mapping: Given the components of a vector, v^{i_1} at point P_1, we obtain the components v^{i_2} of the parallel transported vector at point P_2 by solving equation (15). It is convenient to define a two-point tensor, $g^{i_2}_{\ j_1}$, called the parallel propagator [16], which gives the components of a vector under parallel translation of the vector from point P_1 to point P_2. Given a vector with components v^{i_1} at point P_1, the propagator $g^{i_2}_{\ j_1}$ relates the components of this vector at P_1 to the components v^{i_2} of this same vector after parallel translation to point P_2

$$v^{i_2} = g^{i_2}_{\ i_1} v^{i_1} \tag{16}$$

In a general Riemannian space, the components of the vector at point P_2 depend on the path of parallel translation from P_1 to P_2, in the sense that the path must be a geodesic by the definition of the parallel propagator. However, in Euclidean space these components are completely path independent; the components depend only on the end points P_1 and P_2.

A vector is considered as a geometric object, which means that it is independent of coordinate system. In a Riemannian space, under the operation of parallel propagation a vector changes in such a way that its magnitude stays the same but its absolute direction can change because of the curvature of the space [30]. The direction of the parallel propagated vector is, of course, referred to the local basis vectors. That the vector direction changes under parallel translation can be understood by taking a vector at point P and parallel translating it over a curve that returns to point P. When compared at point P, the components of the initial vector and the round-trip-parallel-transported vector will (in general) be different. It is in this sense that a vector changes its direction under parallel transport.

As mentioned above, the change in the vector that results under parallel transport depends on the path of parallel propagation (a geodesic). Two vectors that are parallel propagated along the same path will maintain the angle between them along the path.

In a Euclidean space, a vector (the geometric object) is considered to be unchanged when parallel propagated. The only thing that happens is that the components of the vector on the local basis must change according to what is required to keep the vector "pointing in the same direction."

In Euclidean space, the parallel propagator in Cartesian coordinates is trivial—its components are just the components of a delta function. The

components of a vector at point P_1 are related to the components of the same vector parallel translated to point P_2 by the propagator (whose components are given in a Cartesian coordinate basis):

$$\delta^{i_2}{}_{j_1} = \begin{cases} +1 & i = j \\ 0 & i \neq j \end{cases} \tag{17}$$

Equation (17) agrees with our notion from elementary geometry that in Cartesian coordinates the vector components are constant under parallel propagation. However, using the parallel propagator in Cartesian coordinates, we can, for example, compute the propagator $g^{i_2}{}_{j_1}$ in curvilinear coordinates $x^i = (r, \theta, \phi)$ given in equation (5)–(7), by the two-point tensor transformation rule

$$g^{i_2}{}_{j_1} = \frac{\partial\, x^i(P_2)}{\partial\, z^m} \frac{\partial\, z^n(P_1)}{\partial\, x^j} \delta^{m_2}{}_{n_1} \tag{18}$$

The parallel propagator $g^{i_2}{}_{j_1}$ is a two-point tensor because it transforms as a vector under coordinate transformation at point P_1 and under coordinate transformation at point P_2.

In Cartesian coordinates, when the points are made to coincide, $P_2 \to P_1$, the propagator reduces to a Kronecker delta at point P_1: $\delta^{m_2}{}_{n_1} \to \delta^m{}_n(P_1)$. In general curvilinear coordinates, when the points P_1 and P_2 coincide, the parallel propagator reduces to the mixed components of the metric tensor $g^{i_2}{}_{j_1}$:

$$\lim_{P_2 \to P_1} g^{i_2}{}_{j_1} \to g^i{}_j(P_1) \quad \text{(metric at } P_1\text{)} \tag{19}$$

The mixed components of the metric tensor at P_1, $g^i{}_j(P_1) \equiv g^{ik}(P_1)\, g_{kj}(P_1)$, are a Kronecker delta—a unit tensor whose components are the same in all systems of coordinates. Indices can be lowered on two-point tensors using the appropriate metric. For example, the index i of the propagator $g^{i_2}{}_{j_1}$ can be lowered by using the metric tensor at point P_2:

$$g_{k_2 j_1} = g_{ki}(P_2)\, g^{i_2}{}_{j_1} \tag{20}$$

When the points are made to coincide, $P_2 \to P_1$, the covariant components of the propagator become the covariant components of the metric tensor at P_1, $g_{k_2 j_1} \to g_{kj}(P_1)$, where $g_{kj}(P_1)$ is the metric at P_1.

The covariant derivatives of the world function $\Omega(P_1, P_2)$ between points P_1 and P_2 are related to the parallel propagator by [16]

$$\Omega_{i_1 j_1} = g_{i_1 j_1} \quad \text{(metric at } P_1\text{)} \tag{21}$$

$$\Omega_{i_1 j_2} = \Omega_{j_2 i_1} = -g_{i_1 j_2} = -g_{j_2 i_1} \quad \text{(parallel propagator)} \tag{22}$$

$$\Omega_{i_2 j_2} = g_{i_2 j_2} \quad \text{(metric at } P_2\text{)} \tag{23}$$

Other useful properties of the parallel propagator are discussed by Synge [16].

3.3 Position Vector

The position vector occupies a central role in deformation theory. For this reason, I discuss it in detail below. In elementary geometry, a point P can be identified by its position vector $\mathbf{r}$, which can be specified in Euclidean-space Cartesian coordinates as

$$\mathbf{r} = z^n \, \mathbf{i}_n \tag{24}$$

where z^n are the Cartesian components of the vector $\mathbf{r}$ and also the Cartesian coordinates of the point P. In terms of general coordinate basis vectors $\mathbf{e}_n = \partial/\partial x^n$ associated with the curvilinear coordinates x^n, the vector $\mathbf{r}$ is given by

$$\mathbf{r} = z^n \, A^m{}_n(P) \, \mathbf{e}_m(P) = \zeta^m \, \mathbf{e}_m(P) \tag{25}$$

The position vector is a geometric object at point P. Among all basis vectors, the Cartesian basis vectors $\mathbf{i}_n$ are unique in that they are usually not associated with a particular spatial point. However, when we express these Cartesian basis vectors in terms of curvilinear basis vectors $\mathbf{e}_n$, then we must imagine that these basis vectors exist at a particular point P. Hence, the transformation between the Cartesian basis vector $\mathbf{i}_m$ and curvilinear coordinate basis vectors $\mathbf{e}_m$ at point P associated with coordinates x^i is given by

$$\mathbf{i}_n(P) = A^m{}_n(P) \, \mathbf{e}_m(P) \tag{26}$$

where the matrix $A^m_n(P)$ depends on the coordinates of point P:

$$A^m{}_n(P) = \frac{\partial \, x^m}{\partial \, z^n}(P) \tag{27}$$

In Cartesian coordinates, the components of the vector $\mathbf{r}$ are simply the Cartesian coordinates z^n of point P. The three numbers (z^1, z^2, z^3) transform as the components of a vector under orthogonal coordinate transformations. Note that in curvilinear coordinates, the components of the position vector, ζ^m, are not the curvilinear coordinates of point P. Also, note that the position vector $\mathbf{r}$ of point P has a magnitude equal to the Euclidean length from the origin of coordinates, say point O, to point P. The position vector of point P is a geometric object at point P, however, it also depends on the point O. This dependence on point O is coordinate independent. Therefore, the position vector of point P is a two-point tensor; it depends on point P and on point O. The transformation properties of the position vector are that of a scalar when a change of coordinates is made at point O and the transformation is that of a vector when coordinates at point P are changed.

In a Riemannian (a generalization of Euclidean space) space, the components of the position vector $r_i(P)$ at point P can be defined in terms of the covariant derivative of the world function

$$r_{i_P} = \frac{\partial \, \Omega(O,P)}{\partial \, z^i_P} \equiv \Omega_{i_P}(O,P) = [2\,\Omega(O,P)]^{1/2} \, \hat{r}_i(P) \tag{28}$$

where $\hat{r}_{i_P}$ is a unit vector at point P and $[2\,\Omega(O,P)]^{1/2}$ is the length of the geodesic from point O to point P. For Euclidean space, $[2\,\Omega(O,P)]^{1/2}$ is the length of the straight line $\overline{OP}$. Equation (28) shows explicitly that the position vector, r_{i_P}, is a two-point tensor.

3.4 Displacement Vector

Consider an elastic body that undergoes a finite deformation in time. The deformation can be specified by a flow function or displacement mapping function

$$z^k = z^k(Z^m, t) \tag{29}$$

where the coordinates z^k (here taken to be Cartesian) of a particle at point Q at time t are given in terms of the particle's coordinates Z^k of point P in some reference state (configuration) at time $t = t_o$, so that $z^k(Z^m, t_o) = Z^k$. We assume the deformation mapping function has an inverse, which is quoted here for later reference

$$Z^m = Z^m(z^k, t) \tag{30}$$

We assume that both the coordinates z^k and Z^m refer to the same Cartesian coordinate system. *

In deformation theory, the initial position of the particle in the medium at point P is specified by a vector

$$\mathbf{R}(P) = Z^k\,\mathbf{i}_k(P) \tag{31}$$

and the final position is specified by a position vector

$$\mathbf{r}(Q) = z^k\,\mathbf{i}_k(Q) \tag{32}$$

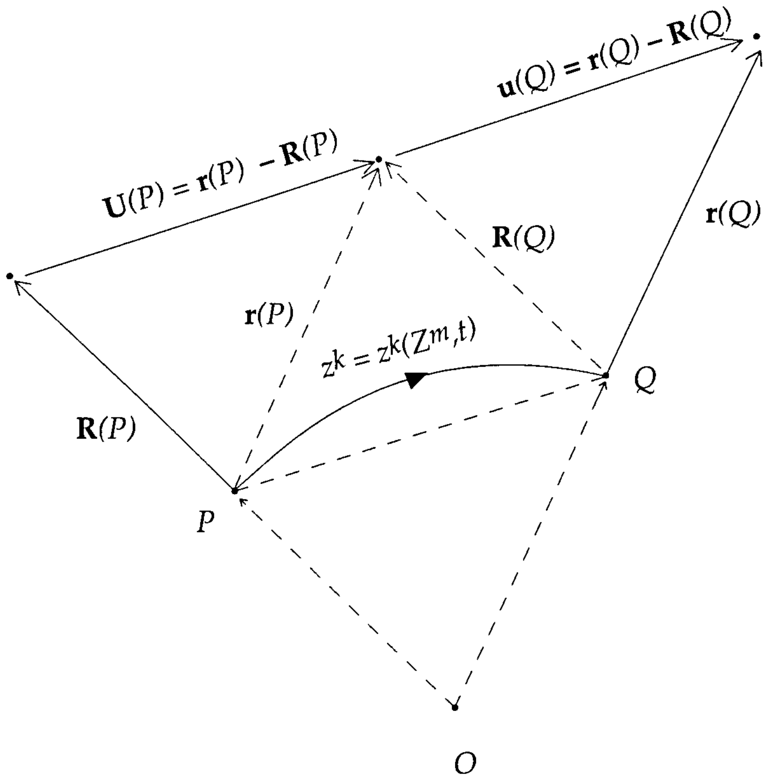
where the quantities $\mathbf{i}_k(P)$ and $\mathbf{i}_k(Q)$ are the Cartesian basis vectors at point P and point Q, respectively. Conventionally, the deformation of a medium is described by specifying the displacement "vector field," which is defined as a difference of these two position vectors. However, the basis vectors $\mathbf{i}_k(P)$ and $\mathbf{i}_k(Q)$ are at different points in the space. Since vectors can be subtracted only if they are at the same point, we must parallel translate $\mathbf{i}_k(P)$ to point Q, or, parallel translate $\mathbf{i}_k(Q)$ to point P. Depending on which mapping we choose, we arrive at the Eulerian or the Lagrangian displacement vector.

First, we parallel translate vector $\mathbf{R}(P)$ to point Q, and then subtract the vectors at point Q (see fig. 2). This procedure defines the components of the Eulerian displacement vector at point Q:

$$\mathbf{u}(Q) = \mathbf{r}(Q) - \mathbf{R}(Q) \tag{33}$$

*I use the notation z^k and Z^k for the Cartesian coordinates and also for the deformation mapping functions. There is no chance of confusing these two since the context makes it clear which one is used in a given instant.

Figure 2. The initial and final position vectors, $\mathbf{R}(P)$ and $\mathbf{r}(Q)$, respectively, are shown as well as their respective parallel translated vectors, $\mathbf{R}(P)$ and $\mathbf{r}(Q)$. Also, shown by a solid curve is the actual displacement path of a representative particle of the medium, labeled by $z^k = z^k(Z^m, t)$. The dashed straight line is the line (geodesic) connecting the initial and final particle positions. The Eulerian displacement vector is $\mathbf{u} = \mathbf{r}(Q) - \mathbf{R}(Q)$, which is the difference of two position vectors at point Q. The Lagrangean displacement vector, $\mathbf{U} = \mathbf{r}(P) - \mathbf{R}(P)$, is the difference of two position vectors at point P.



This Eulerian displacement vector in equation (33) is often called the displacement vector in the spatial representation [8,9,31]. Alternatively, we can parallel translate the vector $\mathbf{r}(Q)$ to point P, and then subtract the vectors at point P. This procedure defines the Lagrangian displacement vector at point P:

$$\mathbf{U}(P) = \mathbf{r}(P) - \mathbf{R}(P) \tag{34}$$

This Lagrangian displacement vector in equation (34) is often called the displacement vector in the material representation [8,9,31]. Equations (33) and (34) show that these two vectors are actually referred to basis vectors at different points. In fact, the two vectors $\mathbf{u}(Q)$ and $\mathbf{U}(P)$ are related by parallel translation. In a Euclidean space, these vectors are the same geometric objects but they are expressed in terms of basis vectors located at different positions P and Q (see below).

In order to further clarify the transformation properties of these two displacement vectors, we use the position vector as discussed in the previous section. Consider the deformation mapping function in curvilinear coordinates, $x^k(X^m, t)$. This function specifies the coordinates x^k (point Q) of a particle at current time t in terms of the coordinates X^k (point P) of the particle in the reference configuration at time $t = t_o$, so that

$$x^k(X^m, t_o) = X^k \tag{35}$$

In addition, there exists a straight line (a geodesic) Γ connecting the points P and Q.

The covariant components of the position vector of point P, $\mathbf{R}(P) = R^n\, \mathbf{e}_n(P)$, in curvilinear coordinates x^i are given by (see equation (28))

$$R_{i_P} = \frac{\partial\, \Omega(O,P)}{\partial\, x^i_P} \equiv \Omega_{i_P}(O,P) = [2\;\Omega(O,P)]^{1/2}\; \hat{R}_{i_P} \tag{36}$$

where $\hat{R}_{i_P}$ are the components of the unit vector at point P tangent to Γ that connects point P and Q. Similarly, the covariant components of vector $\mathbf{r}(Q) = r^n\, \mathbf{e}_n(Q)$ in curvilinear coordinates x^i are given by

$$r_{i_Q} = \frac{\partial\, \Omega(O,Q)}{\partial\, x^i_Q} \equiv \Omega_{i_Q}(O,Q) = [2\, \Omega(O,Q)]^{1/2}\; \hat{r}_{i_Q} \tag{37}$$

where $\hat{r}_{i_Q}$ are the components of the unit vector at point Q tangent to Γ. From equations (36) and (37) it is clear that both R_{i_P} and r_{i_Q} are two-point tensor objects. The quantity R_{i_P} depends on point O and P and transforms as a vector under coordinate transformations at P and as a scalar under coordinate transformations at point O. The quantity r_{i_Q} transforms as a vector under coordinate transformations at point Q and as a scalar under coordinate transformations at point O.

The components of the Eulerian displacement vector in equation (33) (at point Q) are defined in terms of the components of R^{i_P} parallel translated to point Q:

$$\begin{aligned} R^{i_Q} &= g^{i_Q}{}_{j_P}\, R^{j_P} && (38)\\ &= g^{i_Q}{}_{j_P}\; \Omega^{j_P}(O,P) && (39) \end{aligned}$$

where $\Omega^{j_P}(O,P) = g^{jk}(P)\, \Omega_{k_P}(O,P)$, and $g^{jk}(P)$ is the metric at point P in coordinates x^i. The contravariant components of the Eulerian displacement vector in equation (33) are given by

$$u^{i_Q} = \Omega^{i_Q}(O,Q) - g^{i_Q}{}_{j_P}\; \Omega^{j_P}(O,P) \tag{40}$$

Similarly, the components of the Lagrangian displacement vector are given by

$$U^{i_P} = g^{i_P}{}_{j_Q}\; \Omega^{j_Q}(O,Q) - \Omega^{i_P}(O,P) \tag{41}$$

The vectors whose components are U^{i_P} and u^{i_Q} are related by parallel transport along the geodesic Γ connecting P and Q (not along the particle displacement line given by equation (29)). Transporting U^{i_P} to point Q

$$\begin{aligned} U^{i_Q} &= g^{i_Q}{}_{k_P}\; U^{k_P} && (42)\\ &= g^{i_Q}{}_{j_P}\; \left[g^{j_P}{}_{k_Q}\; \Omega^{k_Q}(O,Q) - \Omega^{j_P}(O,P)\right] && (43)\\ &= \delta^i{}_k(Q)\Omega^{k_Q}(O,Q) - g^{i_Q}{}_{j_P}\; \Omega^{j_P}(O,P) && (44)\\ &= u^{i_Q} && (45) \end{aligned}$$

where in the transition from equation (43) to (44) I have used the identity satisfied by the parallel propagator:

$$\delta^i{}_k(Q) = g^{i_Q}{}_{j_P}\, g^{j_P}{}_{k_Q} \tag{46}$$

where $\delta^i{}_k(Q)$ is a unit tensor (delta function) at point Q. Equation (46) is the statement that parallel propagation of a vector has an inverse, so the result that $U^i(P)$ and $u^i(Q)$ are related by parallel transport is true in both Euclidean and Reimannian spaces.

4. Strain Tensors

At time $t = t_o$, consider two particles in the medium that are at positions P_1 and P_2, respectively, and are separated by a finite distance $\Delta S = [2\,\Omega(P_1, P_2)]^{1/2}$. At a later time $t > t_o$ these particles have moved to new positions Q_1 and Q_2 and are separated by a distance $\Delta s = [2\,\Omega(Q_1, Q_2)]^{1/2}$ (see fig. 3).

As a measure of strain, I take the 4-point scalar

$$\Psi(P_1, P_2; Q_1, Q_2) \equiv (\Delta s)^2 - (\Delta S)^2 = 2\Omega(Q_1, Q_2) - 2\Omega(P_1, P_2) \qquad (47)$$

Note that $\Psi(P_1, P_2; Q_1, Q_2)$ depends on four points P_1, P_2, Q_1, and Q_2, and by virtue of its definition in terms of the world function, Ψ is a true four-point scalar invariant under separate coordinate transformations at each of these four points. In Cartesian coordinates, equation (47) becomes

$$\Psi(z_1^i, z_2^i; Z_1^j, Z_2^j) = \delta_{ij}\,(z_1^i - z_2^i)\,(z_1^j - z_2^j) - \delta_{ij}\,(Z_1^i - Z_2^i)\,(Z_1^j - Z_2^j) \qquad (48)$$

where $z_1^i = z^i(Z_1^m, t)$ and $z_2^i = z^i(Z_2^m, t)$ and they are related to the reference configuration at $t = t_o$ by

$$z^i(Z_1^m, t_o) = Z_1^i \qquad (49)$$

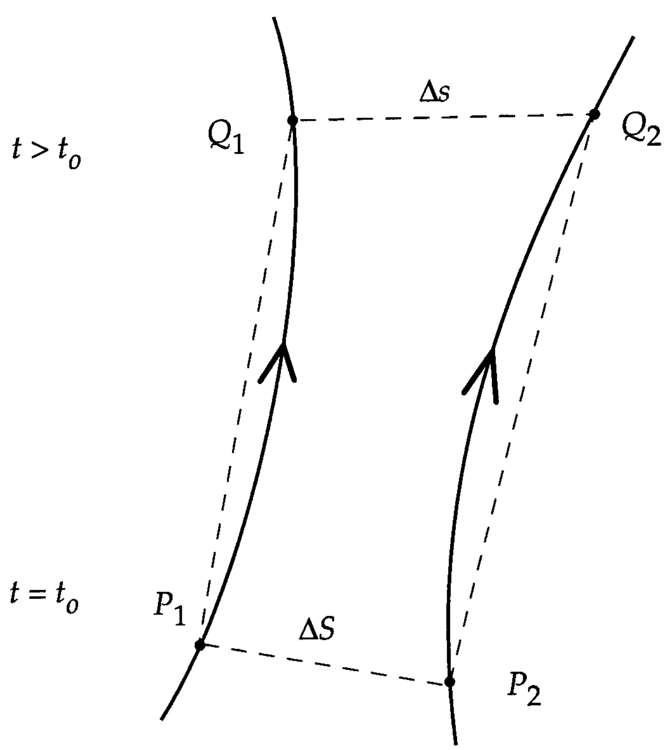


Figure 3. The position P_1 and P_2 of two particles is shown in the reference configuration at $t = t_o$, and the positions Q_1 and Q_2 of the same two particles is shown at later time $t > t_o$. The path of the particles is shown in solid lines and their displacement is shown in dashed lines.

and

$$z^i(Z_2^m, t_o) = Z_2^i \tag{50}$$

and $z^i(Z^m, t)$ is the deformation mapping function in Cartesian coordinates, given in equation (29). If the particle positions P_1 and P_2 are considered as separated by an infinitesimal distance, then, by assuming continuity in the medium and a finite time $t - t_o$, the points Q_1 and Q_2 are also infinitesimally separated. However, because $t - t_o$ is finite, the distance between P_1 and Q_1 is finite (not infinitesimal). Expanding equation (50) about the initial position of the first particle

$$z_2^i = z^i(Z_1^k, t) + \frac{\partial z^i}{\partial Z^j}(Z_1^m, t)\,(Z_2^j - Z_1^j) + \cdots \tag{51}$$

using $z_1^i = z^i(Z_1^k, t)$, leads to the relation between spatial (Eulerian) coordinates z^i and material (Lagrangian) coordinates Z^k

$$\Delta z^i = \frac{\partial z^i}{\partial Z^j}(Z_1^m, t)\,\Delta Z^j + \cdots \tag{52}$$

where $\Delta z^i = z_2^i - z_1^i$ and $\Delta Z^i = Z_2^i - Z_1^i$. Using equation (52) in equation (48) leads to the measure of strain in Cartesian coordinates

$$(\Delta s)^2 - (\Delta S)^2 = \left(\delta_{ij}\frac{\partial z^i}{\partial Z^m}\frac{\partial z^i}{\partial Z^n} - \delta_{mn}\right)\Delta Z^m\,\Delta Z^n + \cdots \tag{53}$$

$$= 2\,E_{mn}\,\Delta Z^m\,\Delta Z^n + \cdots \tag{54}$$

where the quantity in parenthesis is the Lagrangean strain tensor, E_{mn}. The higher order terms in ΔZ^m are small because we can always choose the two initial points P_1 and P_2 arbitrarily close together. From equations (53) and (54), it is clear that the Lagrangean strain tensor is a two-point tensor, depending on initial point P (in the reference configuration at $t = t_o$) and point Q (at time t). Note that in equation (54) there is no restriction to short times $t - t_o$, since we can always choose ΔZ^n sufficiently small.

The Eullerian tensor can be obtained from equation (54) by using the fact that the flow function in equation (29) has an inverse. Using

$$\Delta Z^n = \frac{\partial Z^n}{\partial z^i}(z^i, t)\,\Delta z^i \tag{55}$$

We get the measure of strain in terms of the Eulerian strain tensor e_{mn}:

$$(\Delta s)^2 - (\Delta S)^2 = \left(\delta_{mn} - \delta_{kl}\frac{\partial Z^k}{\partial z^m}\frac{\partial Z^l}{\partial z^n}\right)\Delta z^m\,\Delta z^n + \cdots \tag{56}$$

$$= 2\,e_{mn}\,\Delta z^m\,\Delta z^n + \cdots \tag{57}$$

4.1 Strain Tensor Derivation in Curvilinear Coordinates

I return to the definition of the measure of strain given in equation (47). In the reference configuration at $t = t_o$, consider two particles at points P_1

and P_2 with curvilinear coordinates X_1 and X_2. At a later time t, these two particles are at positions Q_1 and Q_2 with curvilinear coordinates x_1 and x_2, respectively. Consider the first term on the right side of equation (47), which is an integral along a straight line $\overline{Q_1 Q_2}$:

$$\Omega(Q_1, Q_2) = \frac{1}{2}(w_2 - w_1) \int_{w_1}^{w_2} g_{ij} U^i U^j \, dw \tag{58}$$

where $U^i = dx^i(w)/dw$ and where the geodesic (straight line) is parametrized by $x^i(w)$, with $w_1 \leq w \leq w_2$ and the end points are given by $x_1 = x^i(w_1)$ and $x_2 = x^i(w_2)$ (see fig. 3). The function $x^i(w)$ is a solution of the geodesic equation (1). In the case of Eulidean space, and assuming points P_1 and P_2 are arbitrarily close, the geodesic in equation (58) is a straight line given by

$$x^i(w) = x_1^i + \frac{w - w_1}{w_2 - w_1} (x_2^i - x_1^i) \tag{59}$$

The flow function in equation (29) maps the points P_1 and P_2 into the points Q_1 and Q_2. The points Q_1 and Q_2 depend on time t. With reasonable continuity assumptions, and the straight line given in equation (59) with $U^{iQ} = k(x_2^i - x_1^i) = k\,\Delta x^i$ and $k = (w_2 - w_1)^{-1}$, the world function in equation (58) can be approximated by

$$\begin{aligned} \Omega(Q_1, Q_2) &= \frac{1}{2}(w_2 - w_1)\, g_{ij}(Q_1)\, k\Delta x^i \; k\Delta x^j \int_{w_1}^{w_2} dw && (60) \\ &= \frac{1}{2}\, g_{ij}(Q_1)\, \Delta x^i \, \Delta x^j && (61) \end{aligned}$$

Similarly, the second term on the right side of equation (47) can be approximated as

$$\Omega(P_1, P_2) = \frac{1}{2}\, g_{ij}(P_1)\, \Delta X^i \, \Delta X^j \tag{62}$$

where the coordinates X^i are the undeformed ones and $\Delta X^i = X_2^i - X_1^i$. The measure of strain in equation (47) is then

$$(\Delta s)^2 - (\Delta S)^2 = g_{ij}(Q)\, \Delta x^i \; \Delta x^j \; - g_{ij}(P)\, \Delta X^i \; \Delta X^j \tag{63}$$

or using the flow function in equation (29),

$$(\Delta s)^2 - (\Delta S)^2 = \left(g_{ij}(Q)\, \frac{\partial x^i}{\partial X^k}(P,Q)\, \frac{\partial x^j}{\partial X^l}(P,Q) - g_{kl}(P) \right) \Delta X^k \, \Delta X^l \tag{64}$$

Note that x^i and X^i refer to the same system of coordinates. I have dropped the subscripts on Q and P since Q_1 and Q_2, and P_1 and P_2 are infinitesimally close, respectively. The quantity in parenthesis on the right side of equation (64) is twice the Lagrangian strain tensor:

$$2\, E_{k_P l_P} = g_{ij}(Q)\, \frac{\partial x^i}{\partial X^k}(P,Q)\, \frac{\partial x^j}{\partial X^l}(P,Q) \; - g_{kl}(P) \tag{65}$$

From equation (65), it is clear that the Lagrangian strain tensor is a two-point tensor. Under transformation of coordinates at point P, $E_{k_P l_P}$ is a

second rank tensor, while under transformation of coordinates at point Q, it is a scalar. The deformation gradient, $\partial x^i / \partial X^k$, is itself a two-point tensor, as can be seen by its transformation property when coordinates at P and Q are changed.

It is interesting to note that the Lagrangian strain tensor E_{kl} is conventionally thought to be a function of material coordinates, X^i, which coincide with the point P (in the reference configuration) [8,9]. The tensor $E_{k_P l_P}$ can be taken to be a function of only the material coordinates by using the flow mapping function in equation (29), which provides a mapping between all points P and their images, points Q, under the deformation. However, I do not pursue this interpretation below.

The first term in equation (65) is the Green deformation tensor:

$$C_{kl} = g_{ij}(Q)\, \frac{\partial x^i}{\partial X^k}(P,Q)\, \frac{\partial x^j}{\partial X^l}(P,Q) \tag{66}$$

The point P is in the reference configuration at time $t = t_o$ and the point Q is in the deformed state at time t. In equation (66), the Green deformation tensor is naturally a second rank tensor with respect to material coordinate transformations (point P) and it is a scalar with respect to spatial coordinate transformations (point Q). However, the Green tensor is conventionally taken [8,9] as a function of material coordinates by using the flow mapping function in equation (29).

Returning to equation (64), and using the flow mapping function in equation (29), we can obtain

$$(\Delta s)^2 - (\Delta S)^2 = \left(g_{ij}(Q) - g_{kl}(P)\, \frac{\partial X^k}{\partial x^i}(P,Q)\, \frac{\partial X^l}{\partial x^j}(P,Q) \right) \Delta x^i\, \Delta x^j \tag{67}$$

where the Eulerian strain tensor $e_{i_Q j_Q}$ is given by

$$2\, e_{i_Q j_Q} = g_{ij}(Q) - g_{kl}(P)\, \frac{\partial X^k}{\partial x^i}(P)\, \frac{\partial X^l}{\partial x^j}(P) \tag{68}$$

The Eulerian strain tensor $e_{i_Q j_Q}$ is a two-point tensor that is second rank with respect to spatial coordinates at point Q and is a scalar with respect to material coordinates at point P. Once again, however, using the flow mapping function in equation (29), $e_{i_Q j_Q}$ can be imagined to depend on the material coordinates x^i (point Q) of the deformed state.

5. Strain Tensors in Terms of the Displacement Field

The Eulerian and Lagrangian strain tensors can be expressed in terms of the displacement fields u^{i_Q} and U^{i_P} in equations (40) and (41). In terms of covariant components, the displacement field vector at Q is given by

$$u_{i_Q} = \Omega_{i_Q}(O,Q) - g_{i_Q j_P}\,\Omega^{j_P}(O,P) \tag{69}$$

The covariant derivative of u_{i_Q} with respect to point Q (coordinates x^k) is

$$u_{i_Q;k_Q} = \Omega_{i_Q k_Q}(O,Q) - g_{i_Q}{}^{j_P}\,\Omega_{j_P l_P}(O,P)\,\frac{\partial X^l}{\partial x^k} \tag{70}$$

where I used the chain rule for covariant differentiation since $\Omega(O,P) = \Omega(O,P(X))$ and the material coordinates $X^l = X^l(x^k)$ are functions of the spatial coordinates x^k. It is clear that the chain rule must be used in equation (70) by considering Cartesian coordinates. In equation (70), we also used the fact that in Euclidean space the parallel propagator is a constant under covariant differentiation. The notation $\Omega_{i_Q k_Q}(O,Q)$ means the (i,k) component of the second covariant derivative of the world function at point Q. In Euclidean space, these second covariant derivatives are simply related to the parallel propagator (see eqs (21)–(23)).

Using equation (23), the covariant derivative of the displacement field in equation (70) becomes

$$u_{i_Q;k_Q} = g_{ik}(Q) - g_{i_Q}{}^{j_P}\,g_{jl}(P)\,\frac{\partial X^l}{\partial x^k} \tag{71}$$

$$= g_{ik}(Q) - g_{i_Q l_P}\,\frac{\partial X^l}{\partial x^k} \tag{72}$$

where in the last line the metric at P, $g_{jl}(P)$, was used to lower the index on the propagator. Now we multiply equation (72) by the parallel propagator $g^{m_P i_Q}$, sum on i_Q, and solve for the inverse displacement gradient

$$\frac{\partial X^m}{\partial x^k} = g^{m_P}{}_{k_Q} - g^{m_P i_Q}\,u_{i_Q;k_Q} \tag{73}$$

From equation (73), it is clear that in Euclidean space, the deformation gradient $\partial X^m/\partial x^k$ is simply related to the covariant derivative of the displacement field, $u_{i_Q;k_Q}$. Note however, that in a Riemannian space, for finite deformations, it is generally not possible to solve for the deformation gradient [18]. Now, using equation (73) in equation (67), we find the expression relating the two-point Eulerian strain tensor $e_{i_Q j_Q} = e_{i_Q j_Q}(P,Q)$ to the covariant derivatives of the three-point displacement field

$$(\Delta s)^2 - (\Delta S)^2 = \left[u_{i_Q;j_Q} + u_{j_Q;i_Q} - g^{kl}(Q)\,u_{k_Q;i_Q}\,u_{l_Q;j_Q}\right]\Delta x^i\,\Delta x^j \tag{74}$$

$$= 2\,e_{i_Q j_Q}\,\Delta x^i\,\Delta x^j \tag{75}$$

where

$$e_{i_Q j_Q} = \frac{1}{2}\left[u_{i_Q;j_Q} + u_{j_Q;i_Q} - g^{kl}(Q)\, u_{k_Q;i_Q}\, u_{l_Q;j_Q}\right] \tag{76}$$

Equation (76) explicitly shows that the Eulerian strain tensor $e_{i_Q j_Q}$ is a function of two points, material coordinates at point P and spatial coordinates at point Q. Note that $e_{i_Q j_Q}$ is not a function of point O, since by equation (71) the covariant derivative $u_{i_Q;j_Q}$ does not depend on point O. From equation (76) it is also clear that $e_{i_Q j_Q}$ transforms as a second rank tensor under spatial coordinate transformations and that it transforms as a scalar under material coordinate transformations.

An analogous relation can be obtained for the Lagrangian strain tensor by considering the covariant derivative of the displacement field

$$U_{i_P;k_P} = g_{i_P}{}^{j_Q}\, \Omega_{j_Q l_Q}(O,Q) \frac{\partial x^l}{\partial X^k} - \Omega_{i_P k_P}(O,P) \tag{77}$$

Using the relations between the second covariant derivatives of the world function and propagator in equations (21)–(23), and solving for the displacement gradient we get

$$\frac{\partial x^m}{\partial X^k} = g^{m_Q}{}_{k_P} - g^{m_Q i_P}\, U_{i_P;k_P} \tag{78}$$

Inserting the displacement gradient in equation (78) into equation (64) gives an expression for the Lagrangian strain tensor $E_{m_P n_P} = E_{m_P n_P}(P, Q)$,

$$\begin{aligned}(\Delta s)^2 - (\Delta S)^2 &= \left[U_{m_P;n_P} + U_{n_P;m_P} + g^{ij}(P)\, U_{i_P;m_P}\, U_{j_P;n_P}\right] \Delta X^m\, \Delta X^n && (79)\\ &= 2\, E_{m_P n_P}\, \Delta X^i\, \Delta X^j && (80)\end{aligned}$$

where

$$E_{m_P n_P} = \frac{1}{2}\left[U_{m_P;n_P} + U_{n_P;m_P} + g^{ij}(P)\, U_{i_P;m_P}\, U_{j_P;n_P}\right] \tag{81}$$

Equation (79) shows that the Lagrangian strain tensor $E_{m_P n_P}$ is a function of the material coordinates at point P and spatial coordinates at point Q. The Lagrangian strain tensor transforms as a scalar under spatial coordinate transformations at point Q and as a second rank tensor with respect to material coordinate transformations at point P. Note that there are sign differences in equations (76) and (81). Finally, comparing equations (74) and (79), we have the well-known relation between the two strain tensors

$$E_{m_P n_P} = e_{i_Q j_Q} \frac{\partial x^i}{\partial X^m} \frac{\partial x^j}{\partial X^n} \tag{82}$$

Equation (82) provides a complicated relation between the two two-point strain tensors. While the displacement fields u^{i_Q} and U^{i_P} are related to each other by parallel transport (see eqs (40) and (41)), the strain tensors $E_{m_P n_P}$ and $e_{i_Q j_Q}$ are related by two-point deformation gradient tensors, $\partial x^i / \partial X^m$, in equation (82).

6. Summary

Conventionally, the Eulerian and Lagrangian strain tensors are derived either by using shifter tensors or by using convected (moving) coordinates. The definition of the shifter tensor makes use of a scalar product between vectors at two different points in space (without first parallel translating one vector to the position of the other). When convected coordinates are used, vectors and tensors are associated with given coordinates in the convected system of coordinates, rather than being associated with a given point in the underlying space. As discussed in the introduction, both of these features are undesireable when we need to understand the transformation properties of the strain tensors from an inertial frame to a moving frame. These transformation properties are also needed in order to generalize the strain tensor to Riemannian geometry for applications to general relativity.

I have provided a derivation of the Eulerian and Lagrangian strain tensors for finite deformations using the concepts of parallel propagator, the world function of J. L. Synge and the three-point displacement vector field. This derivation avoids the undesireable features mentioned above. The derivation shows that the Eulerian strain tensor is a two-point object that transforms as a scalar under transformation of material coordinates and as a second rank tensor under transformation of spatial coordinates. The derivation also shows that the Lagrangian strain tensor behaves as a scalar under transformation of spatial coordinates and as a second rank tensor under transformation of material coordinates. These transformation properties are useful in understanding how these strain tensors transform from one frame of reference to another moving, non-inertial frame. The formulation presented here of the transformation properties of these tensors is also useful for understanding the role of the reference (unstrained) configuration in prestressed materials, as discussed in the introduction.

Acknowledgments

The author thanks Dr. W. C. McCorkle, U. S. Army Aviation and Missile Command, for suggesting investigation of the problem of stresses in a rotating cylinder, which relied on the ideas in this manuscript as a conceptual foundation.

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE April 2002

3. REPORT TYPE AND DATES COVERED Final, Dec 1999 to April 2000

4. TITLE AND SUBTITLE Transformation Properties of the Lagrangian and Eulerian Strain Tensors

5. FUNDING NUMBERS
DA PR: AH47
PE: 61102A

6. AUTHOR(S) Thomas B. Bahder

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
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Adelphi, MD 20783-1197

8. PERFORMING ORGANIZATION REPORT NUMBER
ARL-TR-908

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Laboratory
Adelphi, MD 20783-1197

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES
ARL PR: 2NENCC
AMS code: 611102.H47

12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*
A coordinate independent derivation of the Eulerian and Lagrangian strain tensors of finite deformation theory is given based on the parallel propagator, the world function, and the displacement vector field as a three-point tensor. The derivation explicitly shows that the Eulerian and Lagrangian strain tensors are two-point tensors, each a function of both the spatial and material coordinates. The Eulerian strain is a two-point tensor that transforms as a second rank tensor under transformation of spatial coordinates and transforms as a scalar under transformation of the material coordinates. The Lagrangian strain is a two-point tensor that transforms as a scalar under transformation of spatial coordinates and transforms as a second rank tensor under transformation of the material coordinates. These transformation properties are needed when transforming the strain tensors from one frame of reference to another moving frame.

14. SUBJECT TERMS Strain, stress, deformation, displacement, tensor, two-point tensor, transformation, Lagrangian, Eulerian, rotation, parallel displacement

15. NUMBER OF PAGES 32

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. Z39-18
298-102